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Operational measure of entanglement based on experimental consequences

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Abstract:
The maximum eigenvalue of the real part of the density matrix expressed in a maximally entangled basis with a particular phase relationship can be used as an operational measure of entanglement. This measure is related to the fidelity, maximized with a local unitary operating on either subsystem, of a standard dense coding, teleportation, or entanglement swapping protocol.

The pure quantum state of two qubits is called entangled if it is impossible to factorize into a tensor product of states for the separate systems (e.g., the singlet state of two spin-$\frac{1}{2}$ particles, $(1/\sqrt{2})(|01\rangle - |10\rangle)$, is entangled) and a mixed state is entangled if it is impossible to represent the density operator as a statistically incoherent sum of factorizable pure states [1]. This property is exploited in the emerging field of *quantum information* [2] in which the nonclassical features of quantum systems are used to surpass classical limitations on communications and computation. For this reason it is desirable to have a physically motivated and mathematically tractable measure of entanglement.

Characterizing the entanglement of a general bipartite quantum system is a difficult problem (See Ref. [3]). Most measures involve difficult extremizations and their physical motivation is not always clear. We take an operational approach examining the fidelities of three closely related quantum information applications: dense coding [4], teleportation [5], and entanglement swapping [6]. These applications are considered with a general two-qubit mixed state in place of the standard maximally entangled pure state, maximizing their operation with local unitaries on the separate sub-systems. We find that these fidelities can be related to maximizing the overlap term between a state ρ and a maximally entangled state $|\Phi\rangle$ with a local unitary operator $\hat{U}$ acting on either subsystem,

$$F_{max} = \max_{\hat{U}}\{\langle\Phi|(\hat{1}\otimes\hat{U})^{\dagger}\hat{\rho}(\hat{1}\otimes\hat{U})|\Phi\rangle\}. \tag{1}$$

This quantity expresses the ability of the state to perform the required quantum information task and is a natural consequence of the fact that one should experimentally attempt to maximize the utility of the state by choosing the best possible local coordinate basis in which to carry out the experiment. This expression can be maximized analytically and used to provide an operational measure of entanglement that has the closed form,

$$E(\rho) = 2\left(\max\{\eta^j, 0\} - \frac{1}{2}\right), \tag{2}$$

where η^j are the eigenvalues of the matrix $M_{n,m} = \mathrm{Re}\{\langle\Phi^n|\hat{\rho}|\Phi^m\rangle\}$, and $|\Phi^i\rangle$ are maximally entangled basis states defined as $\{|\Phi^1\rangle \equiv (|00\rangle + |11\rangle)/\sqrt{2}, |\Phi^2\rangle \equiv i(|01\rangle + |10\rangle)/\sqrt{2}, |\Phi^3\rangle \equiv -(|01\rangle - |10\rangle)/\sqrt{2}, |\Phi^4\rangle \equiv i(|00\rangle - |11\rangle)/\sqrt{2}\}$. In other words, this operational measure is related to the maximum eigenvalue of the real part of the density matrix expressed in this particular Bell basis.

The upper and lower bounds between this operational measure and the concurrence [7] C, the only necessary and sufficient condition for entanglement that has a closed expression for a general two qubit state, are found by a numerical search over a million random density matrices (See Fig 1). These two quantities are equal at

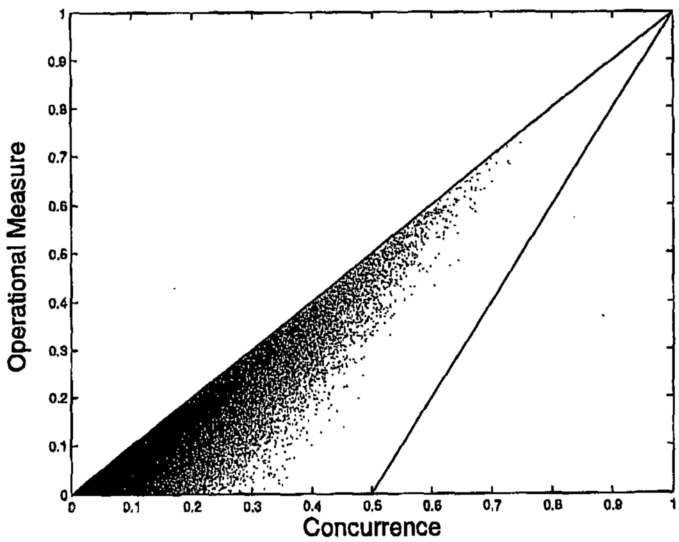

Fig. 1. Operational entanglement vs. concurrence for million random density matrices (We plot 100,000 here for clairty). The upperbound $\{C(\rho_+), E(\rho_+)\}$ and lower $\{C(\rho_-), E(\rho_-)\}$ bounds are given by the solid lines.

the upper bound which occurs for states that are a convex sum of a maximally mixed state and an arbitrary pure state, $\rho_+ = \epsilon\frac{1}{4} + (1-\epsilon)|\psi_{pure}\rangle\langle\psi_{pure}|$, where $(0 < \epsilon < 1)$. If the pure state is decomposed in a Schmidt basis, $|\psi_{pure}\rangle = \left(\hat{U}_A \otimes \hat{U}_B\right)\left(\cos(\theta/2)|00\rangle + \sin(\theta/2)|11\rangle\right)$, the local unitaries will not contribute and we find that $E_+ = C_+ = (1-\epsilon)\sin\theta - \epsilon/2$ (The maximum between this number and zero is implicit). Thus, the operational measure is always less than the concurrence. The lower bound occurs for states that are a convex sum of a direct product state $|uv\rangle$ and a maximally entangled state $|\Phi\rangle$, $\rho_- = \zeta|uv\rangle\langle vu| + (1-\zeta)|\Phi\rangle\langle\Phi|$, such that $\langle uv|\Phi\rangle = 0$ and $(0 < \zeta < 1)$. Taking $|uv\rangle = |01\rangle$ and $|\Phi\rangle = |\Phi^1\rangle$, we compute $E_- = 1 - 2\zeta$ and $C_- = 1 - \zeta$. These bounds, taken together, imply that a non-zero operational entanglement is a necessary condition for entanglement (e.g., nonzero concurrence), but not a sufficient one and they set a range of possible values of concurrence for a state with non-zero operational entanglement,

$$C(\rho_+) \leq E(\rho) \leq C(\rho_-) \tag{3}$$

We also examine the relationship between this operational measure and Bell inequality experiments, which can be used, for example, in quantum cryptography for secure quantum key distribution [8], and we find numerically that this measure is a sufficient condition for violation of a Bell inequality.

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